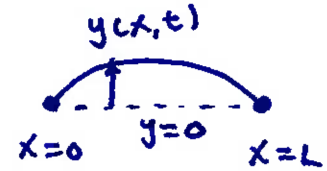
Wave Eq. (part 2)



$y_{tt} = a^2 y_{xx}$ $\quad 0 < x < L \quad t > 0$

$y(0,t) = y(L,t) = 0$ ends fixed

$y(x,0) = f(x)$ initial displacement

$y_t(x,0) = g(x)$ initial velocity

last time : $g(x) = 0$, displacement only ("Problem A")

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a}{L} t\right) \sin\left(\frac{n\pi}{L} x\right)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

today: $f(x) = 0$ initial velocity only ("Problem B")

Same basic idea: $y(x,t) = X(x)T(t)$

$\vdots$

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

two ODEs : $X'' + \lambda X = 0$

$T'' + a^2 \lambda T = 0$

same BCs : $y(0,t) = 0 \rightarrow X(0)T(t) = 0 \rightarrow X(0) = 0$

$y(L,t) = 0 \rightarrow X(L)T(t) = 0 \rightarrow X(L) = 0$

same spatial solution: $\lambda_n = \frac{n^2\pi^2}{L^2} \quad n = 1, 2, 3, \ldots$

$X_n = \sin\left(\frac{n\pi}{L} x\right)$

$$T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0$$

IC: no initial displacement $\rightarrow$ $y(x,0) = 0$

$X(x)T(0) = 0 \rightarrow T(0) = 0$

$$T(t) = A\cos\left(\frac{n\pi a}{L}t\right) + B\sin\left(\frac{n\pi a}{L}t\right)$$

w/ $T(0) = 0 \rightarrow A = 0$

so, $T_n = \sin\left(\frac{n\pi a}{L}t\right)$ (Problem A we had $T_n = \cos\left(\frac{n\pi a}{L}t\right)$)

for each n, $y_n = \sin\left(\frac{n\pi a}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)$

general solution: $\boxed{y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)}$

last IC: $y_t(x,0) = g(x)$

$$y_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} B_n \cos\left(\frac{n\pi a}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$g(x) = \sum_{n=1}^{\infty}\left(\frac{n\pi a}{L}B_n\right)\sin\left(\frac{n\pi}{L}x\right)$$ sine series coeff: $\frac{n\pi a}{L}B_n$

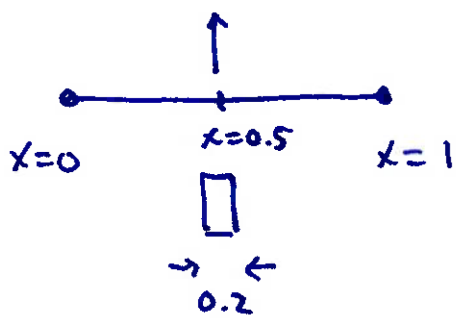
$$\frac{n\pi a}{L}B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$\boxed{B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx}$$

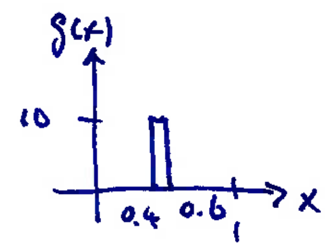
<u>example</u> A string w/ $L=1$, $a=1$ is initially at rest.

It is struck w/ a hammer width 0.2 at $x=0.5$ (center of string)

w/ upward velocity of 10.



initial velocity $g(x) = \begin{cases} 10 & 0.4 < x < 0.6 \\ 0 & \text{else} \end{cases}$

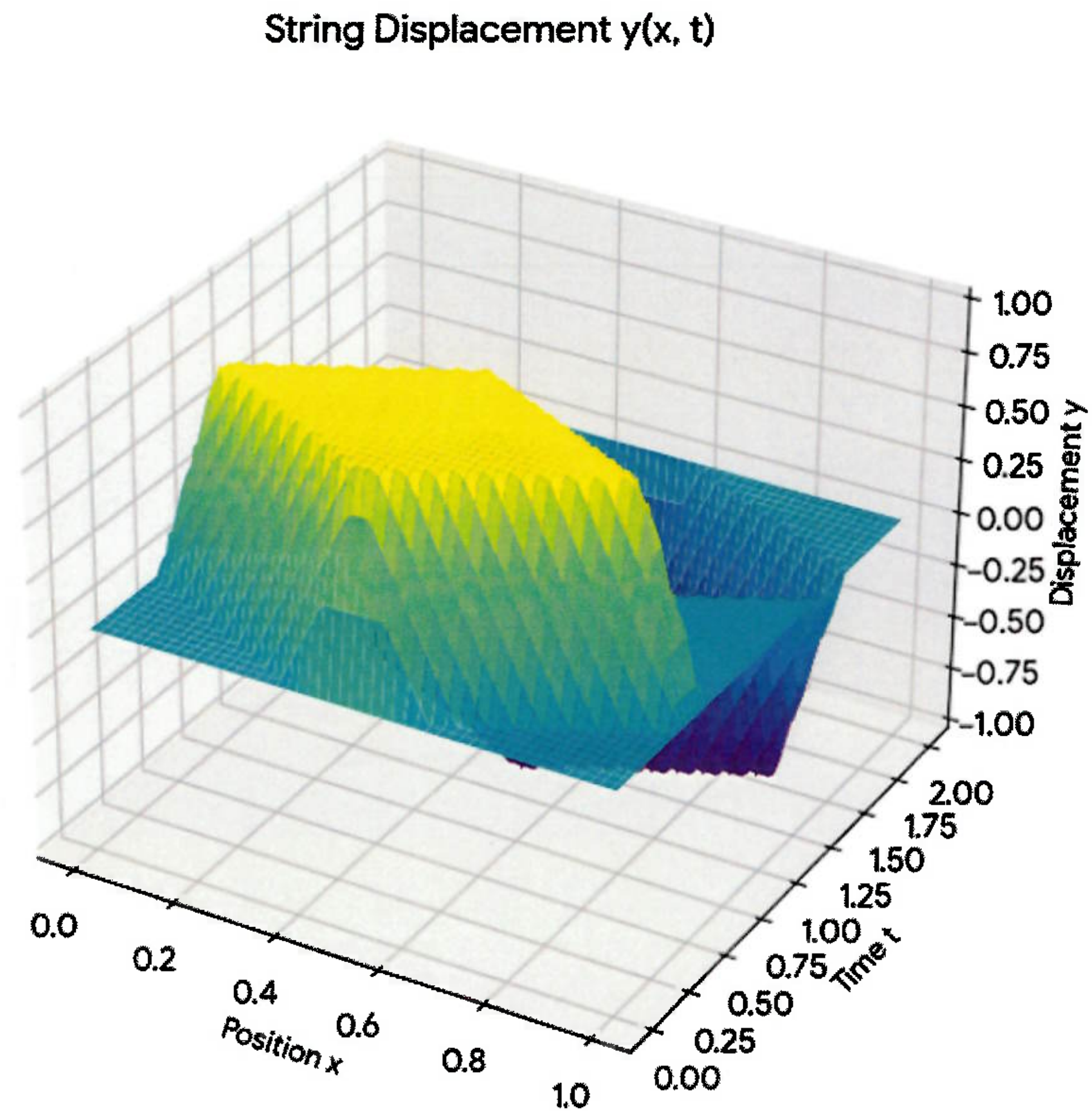


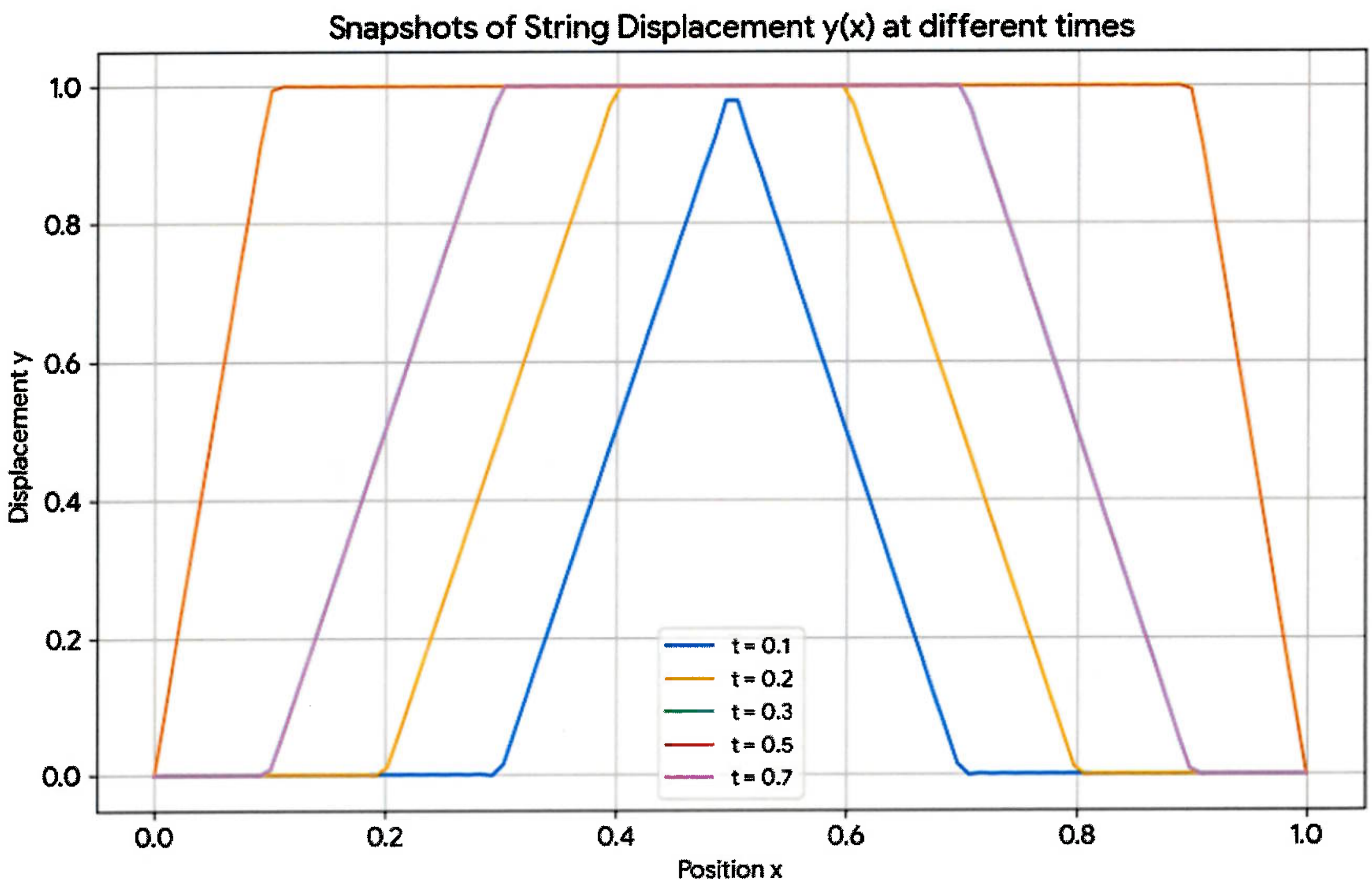
⋮

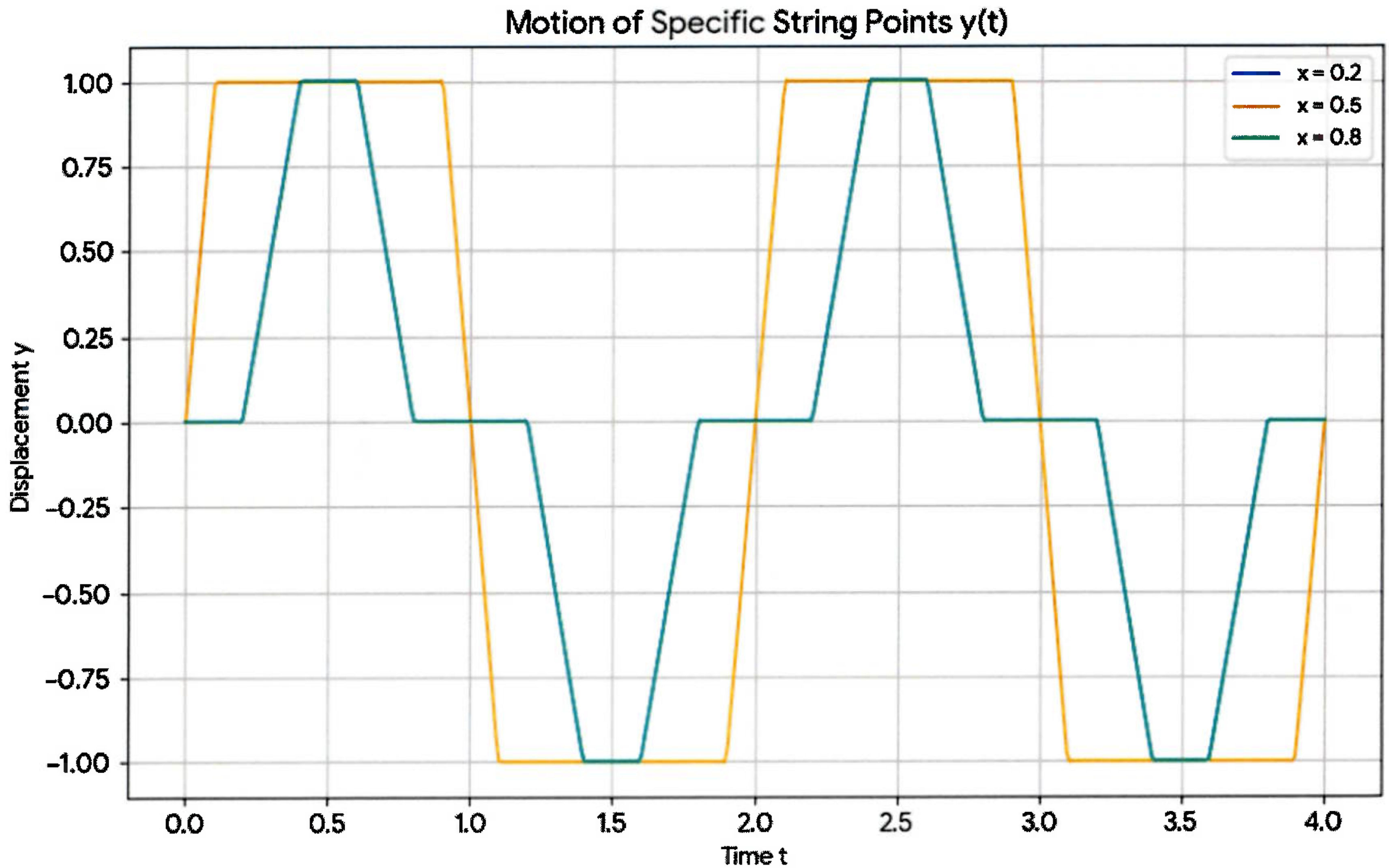
$$y(x,t) = \sum_{n=1}^{\infty} \frac{20}{n^2\pi^2}\left[\cos(0.4n\pi) - \cos(0.6n\pi)\right]\sin(\boxed{n\pi}t)\sin(n\pi x)$$

$\downarrow$

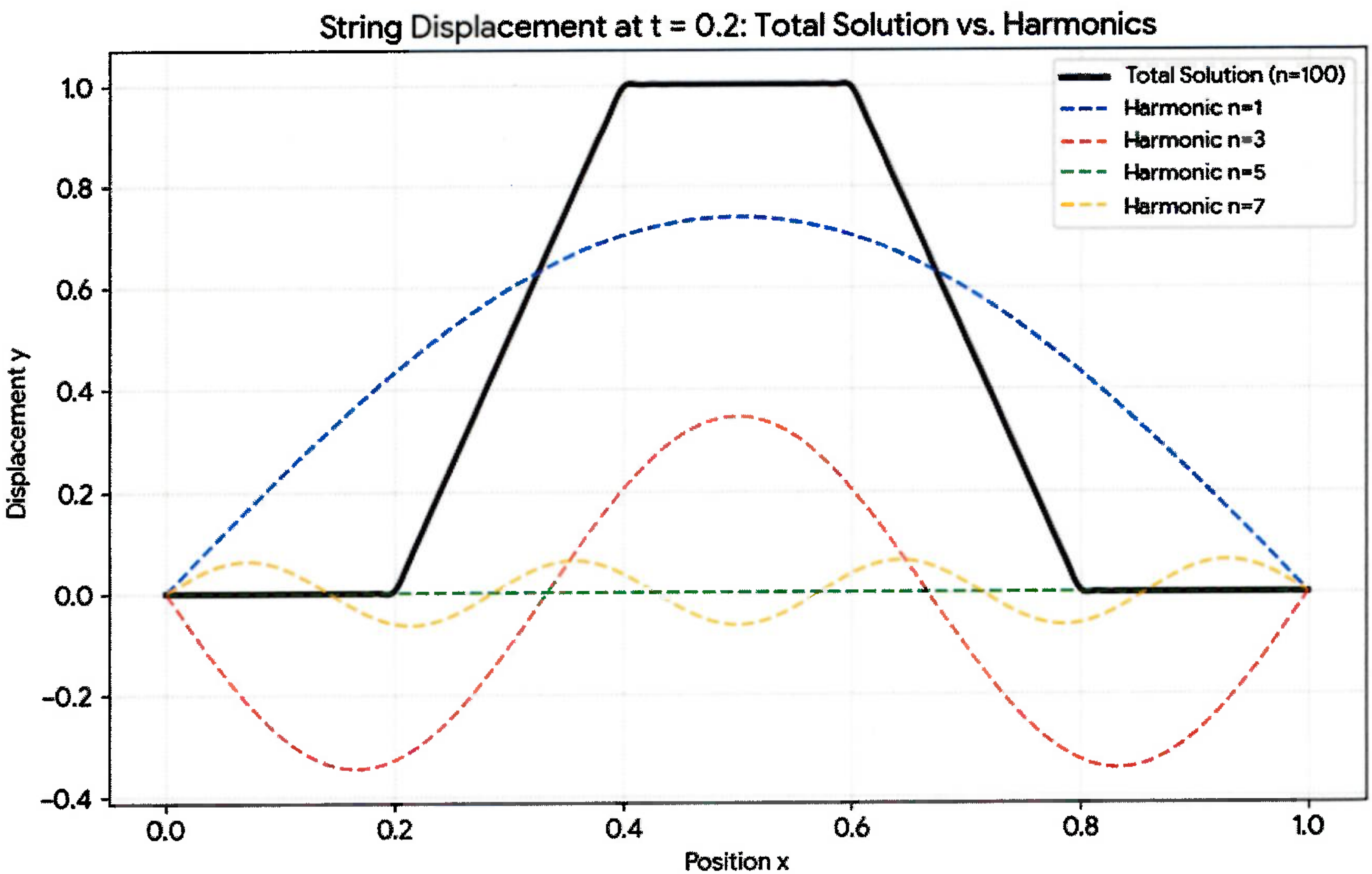
freq. of each mode (n)


String Displacement y(x, t)
Position x
0.0
0.2
0.4
0.6
0.8
1.0
Time t
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
Displacement y
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00


Snapshots of String Displacement y(x) at different times
Displacement y
Position x
0.0
0.2
0.4
0.6
0.8
1.0
t = 0.1
t = 0.2
t = 0.3
t = 0.5
t = 0.7



Motion of Specific String Points y(t)
x = 0.2
x = 0.5
x = 0.8
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
Displacement y
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Time t


String Displacement at t = 0.2: Total Solution vs. Harmonics
Total Solution (n=100)
Harmonic n=1
Harmonic n=3
Harmonic n=5
Harmonic n=7
Displacement y
Position x
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.0
0.2
0.4
0.6
0.8
1.0

the solution we got has no even harmonics (even n's)

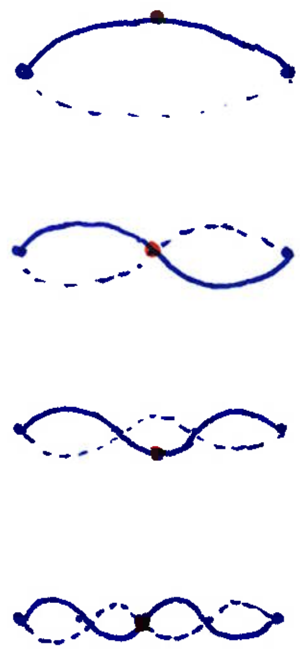
why (physically)?

$n = 1$

$n = 2$

$n = 3$

$n = 4$



we hit the string at $x = 0.5$

→ puts that point in motion

notice for $n =$ even, the center cannot move

hitting center = destroying all even modes

(same if displaced)

on a real piano the impact point is $\frac{L}{7}$ or $\frac{L}{9}$

↓

eliminates all multiples of 7^{th} mode

because 7^{th} and 9^{th} sound dissonant

Problem A: displacement only

Problem B: velocity only

general case: simply add up the solution

because $y_{tt} = a^2 y_{xx}$ is <u>linear</u>

sound from wind instruments $\rightarrow$ pressure waves

eg: $$\frac{\partial^2 p}{\partial t^2} = a^2 \frac{\partial^2 p}{\partial x^2}$$ $p(x,t)$: pressure

same equation!